

# Controllability of Spacecraft Using Only Magnetic Torques

Yaguang Yang\*

July 27, 2015

## Abstract

Spacecraft attitude control using only magnetic torques is a time-varying system. Many designs were proposed using LQR and  $H_\infty$  formulations. The existence of the solutions depends on the controllability of the linear time-varying systems which has not been established. In this paper, we will derive the conditions of the controllability for this linear time-varying systems.

**Keywords:** Spacecraft attitude control, linear time-varying system, reduced quaternion model, controllability.

---

\*Office of Research, NRC, 21 Church Street, Rockville, 20850. Email: yaguang.yang@verizon.net

# 1 Introduction

Spacecraft attitude control using magnetic torque is a very attractive technique because the implementation is simple, the system is reliable (without moving mechanical parts), the torque coils are inexpensive, and their weights are light. The main issue of using only magnetic torques to control the attitude is that the magnetic torques generated by magnetic coils are not available in all desired axes at any time [1]. However, because of the constant change of the Earth's magnetic field as a spacecraft circles around the earth, the controllable subspace changes all the time, many researchers believe that spacecraft's attitude is actually controllable by using only magnetic torques. Numerous spacecraft attitude control designs were proposed in the last twenty five years exploring the features of the time-varying systems [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Some of these papers tried Euler angle model and Linear Quadratic Regulator (LQR) formulations [2, 3, 4, 5, 11] which are explicitly or implicitly assumed that the controllability for the linear time-varying system holds so that the optimal solutions exist [14]. But for the problem of spacecraft attitude control using only magnetic torque, no controllability condition has been established for this linear time-varying system to the best of our knowledge.

Other researchers [6, 7, 8] proposed direct design methods using Lyapunov stabilization theory. The existence of the solutions for these methods implicitly depends on the controllability for the nonlinear time-varying system. Therefore, Bhat [15] investigated controllability of the nonlinear time-varying systems. However, the condition for the controllability of the nonlinear time-varying systems established in this paper is hard to be verified and is a sufficient condition.

Recently, a reduced quaternion model was proposed in [16] and its merits over Euler angle model were discussed in [16, 17, 18]. The reduced quaternion model was also used for the design of spacecraft attitude control system using magnetic torque [10, 12, 13]. Because the controllability of the linear time-varying systems was not established, the existence of the solutions was not guaranteed.

In this paper, we will consider the reduced linear quaternion model proposed in [16] and incorporate control model using only magnetic torques. We will establish the conditions of the controllability for this very general linear time-varying system. The same strategy can easily be used to prove the controllability of the Euler angle based linear time-varying system considered in [5]. However, we will not derive the similar result because of the merits of the reduced quaternion model as discussed in [16, 17, 18].

The remainder of the paper is organized as follows. Section 2 provides a description of the linear time-varying model of the spacecraft attitude control system using only magnetic torque. Section 3 gives the proof of the controllability for this linear time-varying system. The conclusions are summarized in Section 4.

## 2 The linear time-varying model

Let  $\mathbf{J}$  be the inertia matrix of a spacecraft defined by

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \quad (1)$$

We will consider the nadir pointing spacecraft. Therefore, the attitude of the spacecraft is represented by the rotation of the spacecraft body frame relative to the local vertical and local horizontal (LVLH) frame. Therefore, we will represent the quaternion and spacecraft body rate in terms of the rotations of the spacecraft body frame relative to the LVLH frame. Let  $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$  be the body rate with respect to the LVLH frame represented in the body frame,  $\omega_0$  be the orbit (and LVLH frame) rate with respect to the inertial frame, represented in the LVLH frame. Let  $\bar{\mathbf{q}} = [q_0, q_1, q_2, q_3]^T = [q_0, \mathbf{q}^T]^T = [\cos(\frac{\alpha}{2}), \hat{\mathbf{e}}^T \sin(\frac{\alpha}{2})]^T$  be the quaternion representing the rotation of the body frame relative to the LVLH frame, where  $\hat{\mathbf{e}}$  is the unit length rotational axis and  $\alpha$  is the rotation angle about  $\hat{\mathbf{e}}$ . Therefore, the reduced kinematics

equation becomes [16]

$$\begin{aligned} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} \sqrt{1-q_1^2-q_2^2-q_3^2} & -q_3 & q_2 \\ q_3 & \sqrt{1-q_1^2-q_2^2-q_3^2} & -q_1 \\ -q_2 & q_1 & \sqrt{1-q_1^2-q_2^2-q_3^2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \\ &= \mathbf{g}(q_1, q_2, q_3, \boldsymbol{\omega}). \end{aligned} \quad (2)$$

Assume that the inertia matrix of the spacecraft is diagonal which is approximately correct for real systems, let the control torque vector be  $\mathbf{u} = [u_x, u_y, u_z]^T$ , then the linearized nadir pointing spacecraft model with gravity gradient disturbance torque is given as follows [16]:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 \\ f_{41} & 0 & 0 & 0 & 0 & f_{46} \\ 0 & f_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{63} & f_{64} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_x/J_{11} \\ u_y/J_{22} \\ u_z/J_{33} \end{bmatrix} \quad (3)$$

where

$$f_{41} = [8(J_{33} - J_{22})\omega_0^2]/J_{11} \quad (4a)$$

$$f_{46} = (-J_{11} + J_{22} - J_{33})\omega_0/J_{11} \quad (4b)$$

$$f_{64} = (J_{11} - J_{22} + J_{33})\omega_0/J_{33} \quad (4c)$$

$$f_{52} = [6(J_{33} - J_{11})\omega_0^2]/J_{22} \quad (4d)$$

$$f_{63} = [2(J_{11} - J_{22})\omega_0^2]/J_{33}. \quad (4e)$$

The control torques generated by magnetic coils interacting with the Earth's magnetic field is given by (see [1])

$$\mathbf{u} = \mathbf{m} \times \mathbf{b}$$

where the Earth's magnetic field in spacecraft coordinates,  $\mathbf{b}(t) = [b_1(t), b_2(t), b_3(t)]^T$ , is computed using the spacecraft position, the spacecraft attitude, and a spherical harmonic model of the Earth's magnetic field [19]; and  $\mathbf{m} = [m_1, m_2, m_3]^T$  is the spacecraft magnetic coils' induced magnetic moment in the spacecraft coordinates.

The time-variation of the system is an approximate periodic function of  $\mathbf{b}(t) = \mathbf{b}(t+T)$  where  $T = \frac{2\pi}{\omega_0}$  is the orbital period. This magnetic field  $\mathbf{b}(t)$  can be approximately expressed as follows [5]:

$$\begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{bmatrix} = \frac{\mu_f}{a^3} \begin{bmatrix} \cos(\omega_0 t) \sin(i_m) \\ -\cos(i_m) \\ 2 \sin(\omega_0 t) \sin(i_m) \end{bmatrix}, \quad (5)$$

where  $i_m$  is the inclination of the spacecraft orbit with respect to the magnetic equator,  $\mu_f = 7.9 \times 10^{15}$  Wb-m is the field's dipole strength, and  $a$  is the orbit's semi-major axis. The time  $t = 0$  is measured at the ascending-node crossing of the magnetic equator. Therefore, the reduced quaternion linear time-varying system is as follows:

$$\begin{aligned} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 \\ f_{41} & 0 & 0 & 0 & 0 & f_{46} \\ 0 & f_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{63} & f_{64} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_3(t)}{J_{11}} & -\frac{b_2(t)}{J_{11}} \\ -\frac{b_3(t)}{J_{22}} & 0 & \frac{b_1(t)}{J_{22}} \\ \frac{b_2(t)}{J_{33}} & -\frac{b_1(t)}{J_{33}} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \\ &:= \begin{bmatrix} \mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \boldsymbol{\Lambda}_1 & \boldsymbol{\Sigma}_1 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{B}_2(t) \end{bmatrix} \mathbf{m} = \mathbf{A}\mathbf{x} + \mathbf{B}(t)\mathbf{m}. \end{aligned} \quad (6)$$

Substituting (5) into (6) yields

$$\mathbf{B}_2(t) = \begin{bmatrix} 0 & b_{42}(t) & b_{43}(t) \\ b_{51}(t) & 0 & b_{53}(t) \\ b_{61}(t) & b_{62}(t) & 0 \end{bmatrix} \quad (7)$$

where

$$b_{42}(t) = \frac{2\mu_f}{a^3 J_{11}} \sin(i_m) \sin(\omega_0 t) \quad (8a)$$

$$b_{43}(t) = \frac{\mu_f}{a^3 J_{11}} \cos(i_m) \quad (8b)$$

$$b_{53}(t) = \frac{\mu_f}{a^3 J_{22}} \sin(i_m) \cos(\omega_0 t) \quad (8c)$$

$$b_{51}(t) = -\frac{2\mu_f}{a^3 J_{22}} \sin(i_m) \sin(\omega_0 t) = -b_{42} \frac{J_{11}}{J_{22}} \quad (8d)$$

$$b_{61}(t) = -\frac{\mu_f}{a^3 J_{33}} \cos(i_m) = -b_{43} \frac{J_{11}}{J_{33}} \quad (8e)$$

$$b_{62}(t) = -\frac{\mu_f}{a^3 J_{33}} \sin(i_m) \cos(\omega_0 t) = -b_{53} \frac{J_{22}}{J_{33}}. \quad (8f)$$

Therefore, we have

$$b'_{42}(t) = \frac{2\mu_f \omega_0}{a^3 J_{11}} \sin(i_m) \cos(\omega_0 t) \quad (9a)$$

$$b'_{43}(t) = 0 \quad (9b)$$

$$b'_{53}(t) = -\frac{\mu_f \omega_0}{a^3 J_{22}} \sin(i_m) \sin(\omega_0 t) \quad (9c)$$

$$b'_{51}(t) = -\frac{2\mu_f \omega_0}{a^3 J_{22}} \sin(i_m) \cos(\omega_0 t) = -b'_{42} \frac{J_{11}}{J_{22}} \quad (9d)$$

$$b'_{61}(t) = 0 \quad (9e)$$

$$b'_{62}(t) = \frac{\mu_f \omega_0}{a^3 J_{33}} \sin(i_m) \sin(\omega_0 t) = -b'_{53} \frac{J_{22}}{J_{33}} \quad (9f)$$

and

$$b''_{42}(t) = -\frac{2\mu_f \omega_0^2}{a^3 J_{11}} \sin(i_m) \sin(\omega_0 t) \quad (10a)$$

$$b''_{43}(t) = 0 \quad (10b)$$

$$b''_{53}(t) = -\frac{\mu_f \omega_0^2}{a^3 J_{22}} \sin(i_m) \cos(\omega_0 t) \quad (10c)$$

$$b''_{51}(t) = \frac{2\mu_f \omega_0^2}{a^3 J_{22}} \sin(i_m) \sin(\omega_0 t) = -b''_{42} \frac{J_{11}}{J_{22}} \quad (10d)$$

$$b''_{61}(t) = 0 \quad (10e)$$

$$b''_{62}(t) = \frac{\mu_f \omega_0^2}{a^3 J_{33}} \sin(i_m) \cos(\omega_0 t) = -b''_{53} \frac{J_{22}}{J_{33}}. \quad (10f)$$

In matrix format, we have

$$\mathbf{B}'_2(t) = \begin{bmatrix} 0 & b'_{42} & 0 \\ b'_{51} & 0 & b'_{53} \\ 0 & b'_{62} & 0 \end{bmatrix}, \quad (11)$$

and

$$\mathbf{B}''_2(t) = \begin{bmatrix} 0 & b''_{42} & 0 \\ b''_{51} & 0 & b''_{53} \\ 0 & b''_{62} & 0 \end{bmatrix}. \quad (12)$$

A special case is when  $i_m = 0$ , i.e., the spacecraft orbit is on the equator plane of the Earth's magnetic field. In this case,  $\mathbf{b}(t) = [0, -\frac{\mu_f}{a^3}, 0]^T$  is a constant vector. The linear time-varying system of this special case is reduced to a linear time-invariant system whose model is given by

$$\begin{aligned} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 \\ f_{41} & 0 & 0 & 0 & 0 & f_{46} \\ 0 & f_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{63} & f_{64} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b_2/J_{11} \\ 0 & 0 & 0 \\ b_2/J_{33} & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \\ &= \mathbf{Ax} + \mathbf{Bm}. \end{aligned} \quad (13)$$

### 3 The proof of the controllability

The definition of controllability of linear time-varying systems can be found in [20, page 124].

**Definition 3.1** *The linear state equation (6) is called controllable on  $[t_0, t_f]$  if given any  $x_0$ , there exists a continuous input signal  $\mathbf{m}(t)$  defined on  $[t_0, t_f]$  such that the corresponding solution of (6) satisfies  $x(t_f) = 0$ .*

A main theorem used to prove the controllability of (6) is also given in [20, page 127].

**Theorem 3.1** *Let the state transition matrix  $\Phi(t, \tau) = e^{\mathbf{A}(t-\tau)}$ . Denote*

$$\mathbf{K}_j(t) = \frac{\partial^j}{\partial \tau^j} [\Phi(t, \tau) \mathbf{B}(\tau)] \Big|_{\tau=t}, \quad j = 1, 2, \dots \quad (14)$$

*if  $p$  is a positive integer such that, for  $t \in [t_0, t_f]$ ,  $\mathbf{B}(t)$  is  $p$  time continuously differentiable. Then, the linear time-varying equation (6) is controllable on  $[t_0, t_f]$  if for some  $t_c \in [t_0, t_f]$*

$$\text{rank} [\mathbf{K}_0(t_c), \mathbf{K}_1(t_c), \dots, \mathbf{K}_p(t_c)] = n. \quad (15)$$

**Remark 3.1** *If  $\mathbf{A}$  and  $\mathbf{B}$  are constant matrices, the rank condition of (15) for the linear time-varying system is reduced to the rank condition for the linear time-invariant system [20, page 128], i.e., if*

$$\text{rank} [\mathbf{B}, \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}] = n. \quad (16)$$

*then the linear time-invariant system  $(\mathbf{A}, \mathbf{B})$  is controllable.*

First, we consider the special case of (13), the time-invariant system when the spacecraft orbit is on the equator plane of the Earth's magnetic field ( $i_m = 0$ ). Let  $\Sigma$  denote any  $3 \times 3$  diagonal or anti-diagonal matrix with the second row composed of zeros

$$\Sigma := \left\{ \begin{bmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{bmatrix} \right\},$$

and  $\Lambda$  denote any  $3 \times 3$  diagonal matrix with the form

$$\Lambda := \left\{ \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix} \right\}.$$

It is easy to verify that if  $\Sigma_i \in \Sigma$ ,  $\Sigma_j \in \Sigma$ , and  $\Lambda_k \in \Lambda$ , then  $\Sigma_i \Sigma_j \in \Sigma$ ,  $\Sigma_i + \Sigma_j \in \Sigma$ , and  $\Lambda_k \Sigma_i \in \Sigma$ . Using this fact to expand the matrix  $[\mathbf{B}, \mathbf{A}^2\mathbf{B}, \mathbf{A}^3\mathbf{B}, \mathbf{A}^4\mathbf{B}, \mathbf{A}^5\mathbf{B}]$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are defined in (13), shows that the second row of the controllability matrix in (16) is composed of all zeros. This proves that

if the spacecraft orbit is on the equator plane of the Earth's magnetic field, the spacecraft attitude cannot be stabilized by using only magnetic torques.

Now we show that under some simple conditions, the linear time-varying system (6) is controllable for any orbit which is not on the equator plane of the Earth's magnetic field, i.e.,  $i_m \neq 0$ . From (14), we have

$$\begin{aligned} \mathbf{K}_0(t) &= \Phi(t, t)\mathbf{B}(t) = e^{\mathbf{A}(t-t)}\mathbf{B}(t) = \mathbf{B}(t), \\ \mathbf{K}_1(t) &= \frac{\partial}{\partial \tau} [\Phi(t, \tau)\mathbf{B}(\tau)] \Big|_{\tau=t} = \frac{\partial}{\partial \tau} [e^{\mathbf{A}(t-\tau)}\mathbf{B}(\tau)] \Big|_{\tau=t} \\ &= \left[ -\mathbf{A}e^{\mathbf{A}(t-\tau)}\mathbf{B}(\tau) + e^{\mathbf{A}(t-\tau)}\mathbf{B}'(\tau) \right] \Big|_{\tau=t} = -\mathbf{A}\mathbf{B}(t) + \mathbf{B}'(t), \end{aligned} \quad (17)$$

$$\begin{aligned} \mathbf{K}_2(t) &= \frac{\partial^2}{\partial \tau^2} [\Phi(t, \tau)\mathbf{B}(\tau)] \Big|_{\tau=t} \\ &= \left[ \mathbf{A}^2 e^{\mathbf{A}(t-\tau)}\mathbf{B}(\tau) - 2\mathbf{A}e^{\mathbf{A}(t-\tau)}\mathbf{B}'(\tau) + e^{\mathbf{A}(t-\tau)}\mathbf{B}''(\tau) \right] \Big|_{\tau=t} \\ &= \mathbf{A}^2\mathbf{B}(t) - 2\mathbf{A}\mathbf{B}'(t) + \mathbf{B}''(t). \end{aligned} \quad (18)$$

Using the notation of (6), we can rewrite equation (17) as

$$\mathbf{K}_1(t) = - \begin{bmatrix} \mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \Lambda_1 & \Sigma_1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{B}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{B}_2 \\ -\Sigma_1\mathbf{B}_2 + \mathbf{B}_2' \end{bmatrix}.$$

Since

$$\mathbf{A}^2\mathbf{B} = \mathbf{A} \begin{bmatrix} \mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \Lambda_1 & \Sigma_1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \Lambda_1 & \Sigma_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\mathbf{B}_2 \\ \Sigma_1\mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Sigma_1\mathbf{B}_2 \\ \frac{1}{2}\Lambda_1\mathbf{B}_2 + \Sigma_1^2\mathbf{B}_2 \end{bmatrix}$$

and

$$-2\mathbf{A}\mathbf{B}' = -2 \begin{bmatrix} \mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \\ \Lambda_1 & \Sigma_1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} -\mathbf{B}_2' \\ -2\Sigma_1\mathbf{B}_2' \end{bmatrix},$$

equation (18) is reduced to

$$\mathbf{K}_2(t) = \mathbf{A}^2\mathbf{B} - 2\mathbf{A}\mathbf{B}' + \mathbf{B}'' = \begin{bmatrix} \frac{1}{2}\Sigma_1\mathbf{B}_2 - \mathbf{B}_2' \\ \frac{1}{2}\Lambda_1\mathbf{B}_2 + \Sigma_1^2\mathbf{B}_2 - 2\Sigma_1\mathbf{B}_2' + \mathbf{B}_2'' \end{bmatrix}.$$

Hence,

$$\begin{aligned} [\mathbf{K}_0(t), \mathbf{K}_1(t), \mathbf{K}_2(t)] &= [\mathbf{B}(t) \mid -\mathbf{A}\mathbf{B}(t) + \mathbf{B}'(t) \mid \mathbf{A}^2\mathbf{B}(t) - 2\mathbf{A}\mathbf{B}'(t) + \mathbf{B}''(t)] \\ &= \begin{bmatrix} \mathbf{0}_3 & -\frac{1}{2}\mathbf{B}_2 & \frac{1}{2}\Sigma_1\mathbf{B}_2 - \mathbf{B}_2'(t) \\ \mathbf{B}_2 & -\Sigma_1\mathbf{B}_2 + \mathbf{B}_2' & \frac{1}{2}\Lambda_1\mathbf{B}_2 + \Sigma_1^2\mathbf{B}_2 - 2\Sigma_1\mathbf{B}_2' + \mathbf{B}_2'' \end{bmatrix}. \end{aligned} \quad (19)$$

Notice that

$$\begin{aligned} &\text{rank}[\mathbf{K}_0(t), \mathbf{K}_1(t), \mathbf{K}_2(t)] \\ &= \text{rank} \left( \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ -2\Sigma_1 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 & -\frac{1}{2}\mathbf{B}_2 & \frac{1}{2}\Sigma_1\mathbf{B}_2 - \mathbf{B}_2'(t) \\ \mathbf{B}_2 & -\Sigma_1\mathbf{B}_2 + \mathbf{B}_2' & \frac{1}{2}\Lambda_1\mathbf{B}_2 + \Sigma_1^2\mathbf{B}_2 - 2\Sigma_1\mathbf{B}_2' + \mathbf{B}_2'' \end{bmatrix} \right) \\ &= \text{rank} \begin{bmatrix} \mathbf{0}_3 & -\frac{1}{2}\mathbf{B}_2 & \frac{1}{2}\Sigma_1\mathbf{B}_2 - \mathbf{B}_2'(t) \\ \mathbf{B}_2 & \mathbf{B}_2' & \frac{1}{2}\Lambda_1\mathbf{B}_2 + \Sigma_1^2\mathbf{B}_2 - 2\Sigma_1\mathbf{B}_2' + \mathbf{B}_2'' \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \mathbf{0}_3 & -\mathbf{B}_2 & \Sigma_1\mathbf{B}_2 - 2\mathbf{B}_2'(t) \\ \mathbf{B}_2 & \mathbf{B}_2' & \frac{1}{2}\Lambda_1\mathbf{B}_2 + \Sigma_1^2\mathbf{B}_2 - 2\Sigma_1\mathbf{B}_2' + \mathbf{B}_2'' \end{bmatrix}, \end{aligned} \quad (20)$$

$$\begin{aligned} &\Sigma_1\mathbf{B}_2 - 2\mathbf{B}_2'(t) \\ &= \begin{bmatrix} 0 & 0 & f_{46} \\ 0 & 0 & 0 \\ f_{64} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_{42}(t) & b_{43}(t) \\ b_{51}(t) & 0 & b_{53}(t) \\ b_{61}(t) & b_{62}(t) & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & b_{42}' & 0 \\ b_{51}' & 0 & b_{53}' \\ 0 & b_{62}' & 0 \end{bmatrix} \\ &= \begin{bmatrix} f_{46}b_{61}(t) & f_{46}b_{62}(t) - 2b_{42}' & 0 \\ -2b_{51}' & 0 & 2b_{53}' \\ 0 & f_{64}b_{42}(t) - 2b_{62}' & f_{64}b_{43}(t) \end{bmatrix}, \end{aligned} \quad (21)$$

and

$$\begin{aligned}
& \frac{1}{2} \mathbf{A}_1 \mathbf{B}_2 + \mathbf{B}_2''(t) \\
&= \frac{1}{2} \begin{bmatrix} f_{41} & 0 & 0 \\ 0 & f_{52} & 0 \\ 0 & 0 & f_{63} \end{bmatrix} \begin{bmatrix} 0 & b_{42}(t) & b_{43}(t) \\ b_{51}(t) & 0 & b_{53}(t) \\ b_{61}(t) & b_{62}(t) & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_{42}'' & 0 \\ b_{51}'' & 0 & b_{53}'' \\ 0 & b_{62}'' & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2} f_{41} b_{42}(t) + b_{42}'' & \frac{1}{2} f_{41} b_{43}(t) \\ \frac{1}{2} f_{52} b_{51}(t) + b_{51}'' & 0 & \frac{1}{2} f_{52} b_{53}(t) + b_{53}'' \\ \frac{1}{2} f_{63} b_{61}(t) & \frac{1}{2} f_{63} b_{62}(t) + b_{62}'' & 0 \end{bmatrix}, \tag{22}
\end{aligned}$$

we have

$$= \left[ \begin{array}{c|c|c} \mathbf{0}_3 & -\mathbf{B}_2 & \mathbf{\Sigma}_1 \mathbf{B}_2 - 2\mathbf{B}_2'(t) \\ \mathbf{B}_2 & \mathbf{B}_2' & \frac{1}{2} \mathbf{A}_1 \mathbf{B}_2 + \mathbf{B}_2'' \end{array} \right] \\
= \begin{bmatrix} 0 & 0 & 0 & 0 & -b_{42}(t) & -b_{43}(t) & f_{46} b_{61}(t) & f_{46} b_{62}(t) - 2b_{42}' & 0 \\ 0 & 0 & 0 & -b_{51}(t) & 0 & -b_{53}(t) & -2b_{51}' & 0 & 2b_{53}' \\ 0 & 0 & 0 & -b_{61}(t) & -b_{62}(t) & 0 & 0 & f_{64} b_{42}(t) - 2b_{62}' & f_{64} b_{43}(t) \\ 0 & b_{42}(t) & b_{43}(t) & 0 & b_{42}' & 0 & 0 & \frac{1}{2} f_{41} b_{42}(t) + b_{42}'' & \frac{1}{2} f_{41} b_{43}(t) \\ b_{51}(t) & 0 & b_{53}(t) & b_{51}' & 0 & -b_{53}(t) & \frac{1}{2} f_{52} b_{51}(t) + b_{51}'' & 0 & \frac{1}{2} f_{52} b_{53}(t) + b_{53}'' \\ b_{61}(t) & b_{62}(t) & 0 & 0 & b_{62}' & 0 & \frac{1}{2} f_{63} b_{61}(t) & \frac{1}{2} f_{63} b_{62}(t) + b_{62}'' & 0 \end{bmatrix}.$$

To show that this matrix is full rank for some  $t_c$ , we show that there is a  $6 \times 6$  submatrix whose determinant is not zero for  $\omega_0 t_c = \frac{\pi}{2}$ . In view of (8), (9), and (10), for this  $t_c$ , we have

$$b_{53}(t_c) = b_{62}(t_c) = b_{51}'(t_c) = b_{42}'(t_c) = b_{53}''(t_c) = b_{62}''(t_c) = 0. \tag{23}$$

Consider the submatrix composed of the 1st, 2nd, 4th, 5th, 7th, 8th columns, and using (23), we have

$$\begin{aligned}
& \det \begin{bmatrix} 0 & 0 & 0 & -b_{42}(t_c) & f_{46} b_{61}(t_c) & f_{46} b_{62}(t_c) - 2b_{42}' \\ 0 & 0 & -b_{51}(t_c) & 0 & -2b_{51}' & 0 \\ 0 & 0 & -b_{61}(t_c) & -b_{62}(t_c) & 0 & f_{64} b_{42}(t_c) - 2b_{62}' \\ 0 & b_{42}(t_c) & 0 & b_{42}' & 0 & \frac{1}{2} f_{41} b_{42}(t_c) + b_{42}'' \\ b_{51}(t_c) & 0 & b_{51}' & 0 & \frac{1}{2} f_{52} b_{51}(t_c) + b_{51}'' & 0 \\ b_{61}(t_c) & b_{62}(t_c) & 0 & b_{62}' & \frac{1}{2} f_{63} b_{61}(t_c) & \frac{1}{2} f_{63} b_{62}(t_c) + b_{62}'' \end{bmatrix} \\
&= \det \begin{bmatrix} 0 & 0 & 0 & -b_{42}(t_c) & f_{46} b_{61}(t_c) & 0 \\ 0 & 0 & -b_{51}(t_c) & 0 & 0 & 0 \\ 0 & 0 & -b_{61}(t_c) & 0 & 0 & f_{64} b_{42}(t_c) - 2b_{62}' \\ 0 & b_{42}(t_c) & 0 & 0 & 0 & \frac{1}{2} f_{41} b_{42}(t_c) + b_{42}'' \\ b_{51}(t_c) & 0 & 0 & 0 & \frac{1}{2} f_{52} b_{51}(t_c) + b_{51}'' & 0 \\ b_{61}(t_c) & 0 & 0 & b_{62}' & \frac{1}{2} f_{63} b_{61}(t_c) & 0 \end{bmatrix} \\
&= b_{51}(t_c) \det \begin{bmatrix} 0 & 0 & -b_{42}(t_c) & f_{46} b_{61}(t_c) & 0 \\ 0 & 0 & 0 & 0 & f_{64} b_{42}(t_c) - 2b_{62}' \\ 0 & b_{42}(t_c) & 0 & 0 & \frac{1}{2} f_{41} b_{42}(t_c) + b_{42}'' \\ b_{51}(t_c) & 0 & 0 & \frac{1}{2} f_{52} b_{51}(t_c) + b_{51}'' & 0 \\ b_{61}(t_c) & 0 & b_{62}' & \frac{1}{2} f_{63} b_{61}(t_c) & 0 \end{bmatrix} \\
&= -(f_{64} b_{42}(t_c) - 2b_{62}') b_{51}(t_c) \det \begin{bmatrix} 0 & 0 & -b_{42}(t_c) & f_{46} b_{61}(t_c) \\ 0 & b_{42}(t_c) & 0 & 0 \\ b_{51}(t_c) & 0 & 0 & \frac{1}{2} f_{52} b_{51}(t_c) + b_{51}'' \\ b_{61}(t_c) & 0 & b_{62}' & \frac{1}{2} f_{63} b_{61}(t_c) \end{bmatrix} \\
&= -b_{42}(t_c) (f_{64} b_{42}(t_c) - 2b_{62}') b_{51}(t_c) \det \begin{bmatrix} 0 & -b_{42}(t_c) & f_{46} b_{61}(t_c) \\ b_{51}(t_c) & 0 & \frac{1}{2} f_{52} b_{51}(t_c) + b_{51}'' \\ b_{61}(t_c) & b_{62}' & \frac{1}{2} f_{63} b_{61}(t_c) \end{bmatrix} \\
&= -b_{42}(t_c) (f_{64} b_{42}(t_c) - 2b_{62}') b_{51}(t_c) \times \\
& \quad \left[ b_{51} b_{62}' f_{46} b_{61} - b_{42} \left( \frac{1}{2} f_{52} b_{51} + b_{51}'' \right) b_{61} + \frac{1}{2} f_{63} b_{61} b_{42} b_{51} \right]. \tag{24}
\end{aligned}$$

Therefore, in view of Theorem 3.1, the time-varying system is controllable if

$$f_{64}b_{42}(t_c) - 2b'_{62} \neq 0, \quad (25)$$

and

$$b_{51}b'_{62}f_{46}b_{61} - b_{42} \left( \frac{1}{2}f_{52}b_{51} + b''_{51} \right) b_{61} + \frac{1}{2}f_{63}b_{61}b_{42}b_{51} \neq 0. \quad (26)$$

Using (4), (8), (9), (10), and noticing that  $\sin(\omega_0 t_c) = \sin(\frac{\pi}{2}) = 1$ , we have

$$\begin{aligned} & f_{64}b_{42}(t_c) - 2b'_{62} \\ &= \frac{(J_{11} - J_{22} + J_{33})\omega_0}{J_{33}} \frac{2\mu_f}{a^3 J_{11}} \sin(i_m) - 2 \frac{\mu_f \omega_0}{a^3 J_{33}} \sin(i_m) \\ &= \frac{2\mu_f \omega_0 \sin(i_m)}{a^3 (J_{11} J_{33})} (J_{33} - J_{22}), \end{aligned}$$

the first condition (25) is reduced to

$$J_{33} \neq J_{22}. \quad (27)$$

Repeatedly using the same relations, we have

$$\begin{aligned} b_{51}b'_{62}f_{46}b_{61} &= \left( -\frac{2\mu_f}{a^3 J_{22}} \sin(i_m) \right) \left( \frac{\mu_f \omega_0}{a^3 J_{33}} \sin(i_m) \right) \left( \frac{(-J_{11} + J_{22} - J_{33})\omega_0}{J_{11}} \right) \left( -\frac{\mu_f}{a^3 J_{33}} \cos(i_m) \right) \\ &= \frac{2\mu_f^3 \omega_0^2 (-J_{11} + J_{22} - J_{33})}{a^9 J_{11} J_{22} J_{33}^2} \sin^2(i_m) \cos(i_m), \end{aligned} \quad (28)$$

$$\begin{aligned} -b_{42} \left( \frac{1}{2}f_{52}b_{51} + b''_{51} \right) b_{61} &= - \left( \frac{2\mu_f}{a^3 J_{11}} \sin(i_m) \right) \left( \frac{3(J_{33} - J_{11})\omega_0^2}{J_{22}} \left( -\frac{2\mu_f}{a^3 J_{22}} \sin(i_m) \right) + \frac{2\mu_f \omega_0^2}{a^3 J_{22}} \sin(i_m) \right) \\ &\quad \left( -\frac{\mu_f}{a^3 J_{33}} \cos(i_m) \right) \\ &= - \left( \frac{2\mu_f}{a^3 J_{11}} \sin(i_m) \right) \left( \frac{2\mu_f \omega_0^2}{a^3 J_{22}^2} \sin(i_m) (-3J_{33} + 3J_{11} + J_{22}) \right) \\ &\quad \left( -\frac{\mu_f}{a^3 J_{33}} \cos(i_m) \right) \\ &= \frac{4\mu_f^3 \omega_0^2 (-3J_{33} + 3J_{11} + J_{22})}{a^9 J_{11} J_{22}^2 J_{33}} \sin^2(i_m) \cos(i_m), \end{aligned} \quad (29)$$

and

$$\begin{aligned} \frac{1}{2}f_{63}b_{61}b_{42}b_{51} &= \frac{(J_{11} - J_{22})\omega_0^2}{J_{33}} \left( -\frac{\mu_f}{a^3 J_{33}} \cos(i_m) \right) \left( \frac{2\mu_f}{a^3 J_{11}} \sin(i_m) \right) \left( -\frac{2\mu_f}{a^3 J_{22}} \sin(i_m) \right) \\ &= \frac{4\mu_f^3 \omega_0^2 (J_{11} - J_{22})}{a^9 J_{11} J_{22} J_{33}^2} \sin^2(i_m) \cos(i_m). \end{aligned} \quad (30)$$

Combining (28), (29), and (30), we can rewrite (26) as

$$\begin{aligned} & b_{51}b'_{62}f_{46}b_{61} - b_{42} \left( \frac{1}{2}f_{52}b_{51} + b''_{51} \right) b_{61} + \frac{1}{2}f_{63}b_{61}b_{42}b_{51} \\ &= \frac{\mu_f^3 \omega_0^2}{a^9 J_{11} J_{22}^2 J_{33}^2} \sin^2(i_m) \cos(i_m) \\ &\quad (2J_{22}(-J_{11} + J_{22} - J_{33}) + 4J_{33}(-3J_{33} + 3J_{11} + J_{22}) + 4J_{22}(J_{11} - J_{22})) \\ &= \frac{2\mu_f^3 \omega_0^2}{a^9 J_{11} J_{22}^2 J_{33}^2} \sin^2(i_m) \cos(i_m) [J_{22}(J_{11} - J_{22} + J_{33}) - 6J_{33}(J_{33} - J_{11})]. \end{aligned} \quad (31)$$



Therefore, the second condition of (26) is reduced to

$$J_{22}(J_{11} - J_{22} + J_{33}) \neq 6J_{33}(J_{33} - J_{11}). \quad (32)$$

We summarize our main result of this paper as the main theorem of this paper.

**Theorem 3.2** *For the linear time-varying spacecraft attitude control system (6) using only magnetic torques, if the orbit is on the equator plane of the Earth's magnetic field, then the spacecraft attitude is not fully controllable. If the orbit is not on the equator plane of the Earth's magnetic field, and the following two conditions hold:*

$$J_{33} \neq J_{22}, \quad (33a)$$

$$J_{22}(J_{11} - J_{22} + J_{33}) \neq 6J_{33}(J_{33} - J_{11}), \quad (33b)$$

*then the spacecraft attitude is fully controllable by magnetic coils.*

**Remark 3.2** *The controllability conditions include only the spacecraft orbit plane and the spacecraft inertia matrix which are very easy to be verified.*

## 4 Conclusions

In this paper, the controllability of spacecraft attitude control system using only magnetic torques is considered. The conditions of the controllability are derived. These conditions are easier to be verified than previously established conditions.

## References

- [1] M.J. Sidi, Spacecraft Dynamics and Control: A Practical Engineering Approach, Cambridge University Press, Cambridge, UK, 1997.
- [2] K.L. Musser and W.L. Ebert, Autonomous spacecraft attitude control using magnetic torquing only, Proceedings of the Flight Mechanics and Estimation Theory Symposium, NASA Goddard Space Flight Center, Greenbelt, MD, pp. 23-38, 1989.
- [3] M.E. Pittelkau, Optimal periodic control for spacecraft pointing and attitude determination, Journal of Guidance, Control, and Dynamics, 16(6), pp. 1078-1064, 1993.
- [4] R. Wisniewski, Linear time varying approach to satellite attitude control using only electromagnetic actuation, Proceedings of the AIAA Guidance, Navigation, and Control Conference, New Orleans, pp. 243-251, 1997.
- [5] M.L. Paiaki, Magnetic torque attitude control via asymptotic period linear quadratic regulation, Journal of Guidance, Control, and Dynamics, 24(2), pp.386-394, 2001.
- [6] M. Lovera and A. Astolfi, Spacecraft attitude control using magnetic actuators, Automatica, 40, pp. 1405-1414, 2004.
- [7] E. Silani and M. Lovera, Magnetic spacecraft attitude control: a survey and some new results, Control Engineering Practice, 13, pp. 357-371, 2005.
- [8] M. Lovera and A. Astolfi, Global magnetic attitude control of spacecraft in the presence of gravity gradient, IEEE Transactions on Aerospace and Electronic System, 42(3), pp.796-805, 2006
- [9] H. Yan, I.M. Ross, and K.T. Alfried, Pseudospectral feedback control for three-axes magnetic attitude stabilization in elliptic orbits, Journal of Guidance, Control, and Dynamics, 30(4), pp. 1107-1115, 2007.

- [10] T. Pulecchi, M. Lovera, and A. Varga, Optimal discrete-time design of three-axis magnetic attitude control laws, *IEEE Transactions on Control System Technology*, 18(3), pp. 714-722, 2010.
- [11] X. Chen and X. Wu, Model predictive control of cube satellite with magnet-torque, *Proceedings of the 2010 IEEE International Conference on Information and Automation*, pp. 997-1002, Harbin, China, 2010.
- [12] M. Rehanoglu and J.R. Hervas, Three-axis magnetic attitude control algorithm for small satellites, *Proceeding of the 5th International Conference on Recent Advance Technologies*, pp. 897-902, Istanbul, 2011.
- [13] A.M. Zanchettin and M. Lovera,  $H_\infty$  attitude control of magnetically actuated satellite, *Proceedings of 18th IFAC World Congress*, pp. 8479-8484, Milano, Italy, 2011.
- [14] R.E. Kalman, Contributions to the theory of optimal control, *Boletin de la Sociedad Matematica Mexicana*, 5(2), pp. 102-109, 1960.
- [15] S.P. Bhat, Controllability of nonlinear time-varying systems: application to spacecraft attitude control using magnetic actuation, *IEEE Transactions on Automatic Control*, 50(11), pp. 1725-1735, 2005.
- [16] Y. Yang, Quaternion based model for momentum biased nadir pointing spacecraft, *Aerospace Science and Technology*, 14(3), 199-202, 2010.
- [17] Y. Yang, Analytic LQR design for spacecraft control system based on quaternion model, *Journal of Aerospace Engineering*, 25(3), pp. 448-453, 2012.
- [18] Y. Yang, Quaternion based LQR spacecraft control design is a robust pole assignment design, *Journal of Aerospace Engineering*, 27(1), , pp. 168-176. 2014.
- [19] J. Wertz, *Spacecraft Attitude Determination and Control*, Kluwer Academic Publishers, Dordrecht, Holland, 1978.
- [20] W.J. Rugh, *Linear System Theory*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.